Constructive Description Logic, Hybrid-Style

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LOAIT 2010

Motivation

- Pratical and theoretical viewpoint;
- Knownledge is often dynamic and incomplete (refinement);
- In classical $\mathcal{ALC}$, a concept C either includes a given entity or not;
- Constructive notation of truth, context-sensitive (stages of information, D. van Dalen);
- Curry-Howard isomorphism, terms as evidences of proofs;
- Beyond of the standard OWA (SWOA to EOWA, Mendler&Scheele);
- Potential benefit: a programming type system;
- An application: reasoning on legal ontologies (LOAIT 2010);

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Constructives Description Logics

Several possible ways pointed out by (de Paiva, 2006):

- Via constructive first-order logic (IFOL);
- Via constructive modal logic (IML, labelled style, by Simpson);
- Via constructive hybrid logic (IHL, by Brauner and de Paiva);

We follow the last two. Might end up with a best style of constructive DL in terms of foundation and easer implementation.

A language for constructives ALC logics

$$C, D ::= A \mid \bot \mid \top \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid C \sqsubseteq D \mid \exists R.C \mid \forall R.C$$

Non-standard is the $C \sqsubseteq D$ as a concept constructor.

The cALC from Mendler-Scheele

Semantics given by a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \perp^{\mathcal{I}}, \cdot^{\mathcal{I}})$ closed under refinement, i.e., $x \in A^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in A^{\mathcal{I}}$.

$$
\begin{aligned}
\top^{\mathcal{I}} &=_{df} \Delta^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta_c^{\mathcal{I}}.x \preceq y \Rightarrow y \notin C^{\mathcal{I}}\} \\
(C \sqcap D)^{\mathcal{I}} &=_{df} C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &=_{df} C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(C \sqsubseteq D)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta_c^{\mathcal{I}}.(x \preceq y \text{ and } y \in C^{\mathcal{I}}) \Rightarrow y \in D^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta_c^{\mathcal{I}}.x \preceq y \Rightarrow \exists z \in \Delta^{\mathcal{I}}.(y,z) \in R^{\mathcal{I}} \text{ and } z \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta_c^{\mathcal{I}}.x \preceq y \Rightarrow \forall z \in \Delta^{\mathcal{I}}.(y,z) \in R^{\mathcal{I}} \Rightarrow z \in C^{\mathcal{I}}\}
\end{aligned}
$$

$\perp^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ are fallible entities closed under refinement. $\Delta_c^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus \perp^{\mathcal{I}}$ are non-fallible elements.

cALC logic

Motivation

Auditing business data.

Properties

- Not always $\exists R.\bot = \bot$ (non-traditional, fallibility, when $\bot^{\mathcal{I}} = \emptyset$, then axiom $\neg\exists R.\bot$)
- $\exists R.(C \sqcup D) = \exists R.C \sqcup \exists R.D$ (non-traditional, if valid, role filling via R is confluent with refinement)
- Excluded Middle, $C \sqcup \neg C$ is not an axiom (standard)

Why distribution of $\exists$ over $\sqcup$ should fail?

The iALC Logic

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Considerations on Legal Ontologies

What is an Ontology?

- A declarative description of a domain.
- Ontology consistency is mandatory.
- Consistency means absence of contradictions.
- Negation is an essential operator.
- Concretely, an Ontology is a Knowledge Base:
- A set of Logical Assertions on a Domain that aim to describe it completely.

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What does it mean the term "Law"?

- What does count as the "unit of law"? Open question, a.k.a. "The individuation problem".
- (Raz1972) What is to count as one "complete law"?

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Two main (distinct) approaches to the "Individuation problem"

- Taking all (existing) legally valid statements as a whole. This totality is called "the law".
- ▷ Legal Positivism tradition (Kelsen1991). Question: Natural coherence versus Knowledge Management resulted coherence.
- Taking into account all individual legally valid statement as individual laws.
- ▷ Facilitates the analysis of structural relationship between laws, viz. Primary and Secondary Rules.
- **The second seems to be quite adequate to Legal AI.**

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Why we do not consider Deontic Modal Logic?

- Deontic Logic does not properly distinguish between the normative status of a situation from the normative status of a norm (rule). (Valente1995)
- Norms should not have truth-value, they are not propositions. (Kelsen1991)

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Using iALC to formalize Conflict of Laws in Space

A Case Study

Peter and Maria signed a renting contract. The subject of the contract is an apartment in Rio de Janeiro. The contract states that any dispute will go to court in Rio de Janeiro. Peter is 17 and Maria is 20. Peter lives in Edinburgh and Maria lives in Rio.

Using iALC to formalize Conflict of Laws in Space

The valid legal statements (individuals)

Only legally capable individuals have civil obligations: $contract \preceq PeterLegalAge$
$contract \preceq MariaLegalAge$

The concepts and their relationships

BR is the "set" of Brazilian Valid Legal Statments
SC is the "set" of Scottish Valid Legal Statments
PIL_{BR} is the "set" of Private International Law in Brasil
$ABROAD$ is the "set" of VLS abroad Brasil
$LexDomicilium$ is a legal connection:
▷ The pair $\langle PeterLegalAge, PeterLegalAge \rangle$ is in it

Using iALC to formalize Conflict of Laws in Space

The Axioms (Subsumptions)

$$
\begin{array}{l}
MariaLegalAge \in BR \\
PeterLegalAge \in SC \\
contract \preceq PeterLegalAge \\
contract \preceq MariaLegalAge \\
PIL_{BR} \sqsubseteq BR \\
SC \sqsubseteq ABROAD \\
\exists LexDomicilium.SC \sqsubseteq \exists LexDomicilium.ABROAD \\
\exists LexDomicilium.ABROAD \sqsubseteq PIL_{BR} \\
\langle PeterLegalAge, PeterLegalAge \rangle \in LexDomicilium
\end{array}
$$

Using *iALC* semantics, one concludes that: $contract \in BR$. Each legal statement generalizing ($\preceq$) *contract* is in *BR*. Interesting case $PeterLegalAge \in \exists LexDomicilium.SC \sqsubseteq PIL_{BR} \sqsubseteq BR$.

Using iALC to formalize Conflict of Laws in Space

Summary of the Approach

- Individual Legal Valid Statements are the individuals of the universe.
- **Concepts** are Classes of individual laws.
- **Roles** (relationships) between individuals laws denote kinds of Legal Connections
- Subsumptions and Negations are intuitionistically interpreted (iALC)

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Motivation for a SC for iALC

- Proof-theoretical approach;
- Simple Tableaux (without analytical cuts) cannot produce short proofs (polynomially lengthy proofs). Sequent Calculus (SC) (with the cut rule) has short proofs.
- We envisage industrial application problems with explanations requirement.

Labeled Formulas

$$LB \rightarrow \forall R \mid \exists R$$
$$L \rightarrow LB, L \mid \emptyset$$
$$\phi_{lc} \rightarrow {}^{L}\phi_c$$

Each labeled $\mathcal{ALC}$ concept has an straightforward $\mathcal{ALC}$ concept equivalent. For example:

$$\exists R_2.\forall Q_2.\exists R_1.\forall Q_1.\alpha \equiv {}^{\exists R_2, \forall Q_2, \exists R_1, \forall Q_1}\alpha$$

The SC for iALC

Rules over roles quantification

$$\frac{\Delta, {}^{L,\forall R}\alpha \Rightarrow \gamma}{\Delta, {}^{L}(\forall R.\alpha) \Rightarrow \gamma}\ \forall\text{-l} \qquad \frac{\Delta \Rightarrow {}^{L,\forall R}\alpha}{\Delta \Rightarrow {}^{L}(\forall R.\alpha)}\ \forall\text{-r}$$

$$\frac{\Delta, {}^{L,\exists R}\alpha \Rightarrow \gamma}{\Delta, {}^{L}(\exists R.\alpha) \Rightarrow \gamma}\ \exists\text{-l} \qquad \frac{\Delta \Rightarrow {}^{L,\exists R}\alpha}{\Delta \Rightarrow {}^{L}(\exists R.\alpha)}\ \exists\text{-r}$$

The SC for iALC

Boolean rules

$$\frac{\Delta, {}^{\forall L}\alpha, {}^{\forall L}\beta \Rightarrow \gamma}{\Delta, {}^{\forall L}(\alpha \sqcap \beta) \Rightarrow \gamma} \sqcap\text{-l} \qquad \frac{\Delta \Rightarrow {}^{\forall L}\alpha \qquad \Delta \Rightarrow {}^{\forall L}\beta}{\Delta \Rightarrow {}^{\forall L}(\alpha \sqcap \beta)} \sqcap\text{-r}$$

$$\frac{\Delta, {}^{\exists L}\alpha \Rightarrow \gamma \qquad \Delta, {}^{\exists L}\beta \Rightarrow \gamma}{\Delta, {}^{L}(\alpha \sqcup \beta) \Rightarrow \gamma} \sqcup\text{-l} \qquad \frac{\Delta \Rightarrow {}^{\exists L}\alpha}{\Delta \Rightarrow {}^{\exists L}(\alpha \sqcup \beta)} \sqcup\text{-r}$$

$$\frac{\Delta \Rightarrow {}^{\neg L}\alpha}{\Delta, {}^{L}\neg\alpha \Rightarrow} \neg\text{-l} \qquad \frac{\Delta, {}^{\neg L}\alpha \Rightarrow}{\Delta \Rightarrow {}^{L}\neg\alpha} \neg\text{-r}$$

The SC for iALC

Generalization rules

$$\frac{\delta \Rightarrow \gamma}{{}^{+\exists R}\delta \Rightarrow {}^{+\exists R}\gamma}\ \text{prom-}\exists \qquad \frac{\Delta \Rightarrow \gamma}{{}^{+\forall R}\Delta \Rightarrow {}^{+\forall R}\gamma}\ \text{prom-}\forall$$

SC for iALC properties

- Soundness (similar previous one)
- Completeness is a working in progress;
- Cut elimination is a working in progress, analog of Takeuti;
- Complexity analysis, probably (intuition) PSPACE-hard (from ALC and IL complexity).
- Can we have a uniform framework for several DLs?

From Mendler-Scheele

$$\frac{}{\alpha \Rightarrow \alpha} \qquad \frac{}{\bot \Rightarrow \alpha}$$

$$\frac{\Gamma_1 \Rightarrow \alpha \qquad \Gamma_2, \beta \Rightarrow \gamma}{\Gamma_1, \Gamma_2, \alpha \sqsubseteq \beta \Rightarrow \gamma} \sqsubseteq\text{-l} \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{\Gamma \Rightarrow \alpha \sqsubseteq \beta} \sqsubseteq\text{-r}$$

$$\frac{\Gamma, \alpha, \beta \Rightarrow \gamma}{\Gamma, (\alpha \sqcap \beta) \Rightarrow \gamma} \sqcap\text{-l} \qquad \frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow (\alpha \sqcap \beta)} \sqcap\text{-r}$$

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$$\frac{\Gamma, \alpha \Rightarrow \beta}{(\forall R.)\Gamma, \forall R.\alpha \Rightarrow \forall R.\beta} \forall \qquad \frac{\Gamma, \alpha \Rightarrow \beta}{(\forall R.)\Gamma, \exists R.\alpha \Rightarrow \exists R.\beta} \exists$$

From IML

$$\frac{}{\Gamma, x\colon \alpha \Rightarrow\ x\colon \alpha, \Delta} \qquad \frac{}{xRy, \Gamma \Rightarrow\ \Delta, xRy}$$

$$\frac{}{\Gamma, x\colon \bot \Rightarrow\ \Delta}$$

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