On the Computational Complexity of the Intuitionistic Hybrid Modal Logics

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Abstract

This article shows that some decision problems for the Intuitionistic Hybrid Modal Logics IK are PSPACE-complete.

1 Introduction

In [3], a general approach to prove computational complexity of Hybrid Logics is presented. There, it is shown how to obtain, from a formula α, a 2-person game, designed to be polynomially implemented in an Alternating Turing Machine, such that, deciding existence of winning strategy for one of the players is equivalent to decide satisfiability (SAT) of α. This approach is used to show that SAT is in PSPACE, since any polynomial time implementation on an Alternating Turing Machine can be done in ordinary Turing Machine using polynomial space. For Hybrid Modal K, for example, it is possible to conclude PSPACE-completeness of SAT, since K is already PSPACE-hard.

The approach briefly explained above has been applied to Classical Modal Logics. In this article we adapt the approach in order to take care of Intuitionistic Modal Logics. We prove PSPACE-completeness of Intuitionistic Modal K (IK, as presented in [22, 25]). It is worth noting that the authors are not aware of any result on the computational complexity of the logic IK. However, in a series of papers (see [30],[29],[28]), Wolter and Zakharyaschev showed how to embed intuitionistic modal logics into standard modal logics extending the combinations of K and S4 (via extension of Goedel translation of intuitionistic modal logic to S4). Given this translation and known complexity results for standard modal logics, the PSPACE-completeness would follow for IK and $IS4$. In [30], it is shown a validity-preserving translation from $IntK$[1] formulas into ones in a bimodal logic contained in $S4 \oplus K$. The intuition is that, the $S4$ modality takes care of the intuitionistic feature of $IntK$, by means of a Gœdel translation, and K takes care of the ordinary modality $\Box$ of $IntK$. In $IntK$, the $\Diamond$ is defined as $\Diamond\alpha \equiv \neg\Box\neg\alpha$. This (classical ?) definition of $\Diamond$ allowed an easier translation from $IntK$ to the bimodal language than an independent approach to $\Diamond$ would take. Nevertheless, Fisher Servi has extended the Gœdel to a translation of $ExtIntK_{\Box\Diamond}$, an interesting logic that imposes weaker connection between $\Box$ and $\Diamond$, into an extension of the bimodal $S4 \oplus K$. Various, intuitionisic based extensions of $ExtIntK_{\Box\Diamond}$ were studied (see [7, 20, 11, 12, 13, 10, 28, 27]). $MIPC$ ([24]) is a particularly interesting case, for Bull [7] has provided a translation of $MIPC$ into monadic first-order intuitionistic logic. This proves that the computational complexity of SAT for $MIPC$ formulas is PSPACE-hard.

As it can be seen from the above paragraph, extensions of Gœdel translations are quite succesful in helping to provide worst-case analysis for SAT problems in intuitionistic modal logics. However, when we take hybrid logics into account, there is no known translation of Intuitionistic Hybrid Modal Logic ($IHML$) into a proposicional language. In the case of $IHML$, the technique of preserving validity translations seems to be useless.

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[1]The reader is adviced that $IntK$ and IK are not the same logic

This article has two purposes: 1- We show that the technique presented in [1] can be adapted to nontrivial bimodal logics, as it is the case of IK^2, and; 2- We show that this very technique can be used for intuitionistic modal hybrid logic (IHK). We prove that both, IK and IHK are PSPACE-hard, and as both are extensions of PSPACE-complete IPL (Intuitionistic propositional logic), then they are PSPACE-complete, regarding SAT.

Since our version of IHK is an extension of IK, section **??** shows the proof of PSPACE-hardness of IHK, leaving IK's proof of complexity as a corollary. The choice for IHK is justified by the use we make of it in the formalization of legal reasoning (see [16, 8, 15]). The kind of constructive reasoning pointed out to the distribution of $\diamond$ over the $\vee$, and the adoption of the axiom $\diamond\bot \leftrightarrow \bot$. In fact in [15, 16, 8] the legal knowledge is formalized by using an intuitionistic version of the Description Logic (ALC) that is obtained from a sub-logic of IHK by the usual relationship between description logic and hybrid modal logic languages (consult [1] for a detailed view). In the present article we focus only on the fact that IHK is a hydrid intuitionistic logic having IK as its propositional basis. Finally it is important to state that a good reason to work with IK *per se* is the fact that the only thing that we have to add to it in order to have K (the classical one) is the excluded middle law. In section **??**, we advice the reader that the logic IHK is not $IHML$. The latter can be seen as fragment of the intuitionistic first-order logic based on IK. While IHK is defined from an intuitionistic version of the description logic ALC. In this section we argue that $IHML$ is 2EXPTIME-hard. Finally, in section **??** we apply the technique presented in [3] to show that IHK and IK are both PSACE-complete. In the conclusion we discuss how this result is related to TBOX reasoning in $iALC$ and if we extend $iALC$ to ABOX reasoning we go to 2EXPTIME-hard satisfiability. The proof of PSPACE-hardness of IHK that is shown in section **??** was firstly presented in the syntax of $iALC$ and can be found in [9]. In order to show main point defended by this article, namely the power of using the technique developed in [3] to Hybrid logics, we will use part of the proof found in [9] in the syntax of IHK.

2 The Intuitionistic Modal Logic IK

IK was introduced during the 80's in [25, 13, 23]. These modal logics arise from interpreting the usual possible worlds definitions in an intuitionistic meta-theory. See [25] for a quite helpful discussion on the many alternative intuitionistic logics that are possible to define , besides IK. The language of IK is the same of classical logic, namely:

$$\varphi ::= p \mid \bot \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \to \varphi_2 \mid \Diamond\varphi \mid \Box\varphi$$

where p is a propositional symbol in Φ, the propositional language of IK.

A Hilbert calculus for IK is provided, after [23, 25, 13]. It contains all axioms of intuitionistic propositional logic and axioms and rules shown in Figure 1. The calculus implements the syntactical relationship $\Theta \vdash \varphi$, where Θ is a set of formulas.

0.	all substitution instances of theorems of Int. Prop. Logic
1.	$\Box(\varphi_1 \to \varphi_2) \to (\Box\varphi_1 \to \Box\varphi_2)$
2.	$\Diamond(\varphi_1 \to \varphi_2) \to (\Diamond\varphi_1 \to \Box\varphi_2)$
3.	$\Diamond(\varphi_1 \vee \varphi_2) \to (\Diamond\varphi_1 \vee \Diamond\varphi_2)$
4.	$\Diamond\bot \to \bot$
5.	$(\Diamond\varphi_1 \to \Box\varphi_2) \to \Box(\varphi_1 \to \varphi_2)$
MP	If φ_1 and $\varphi_1 \to \varphi_2$ are theorems, φ_2 is a theorem too.
Nec	If φ is a theorem then $\Box\varphi$ is a theorem too.

Figure 1: The IK axiomatization

[2] Simpson, [25], argues that IK is the true intuitionistic analoque of K

The syntactical calculus will not be used directly in this article. It is shown also as a matter to help logicians to figure out what are the general laws in IK. With this calculus we can provide an accountable definition of a consistent set of IK formulas.

Definition 1 *A set of formula Γ is consistent in IK, iff, $\Gamma \not\vdash \bot$*

Instead of the calculus, we can also use the semantical notions on IK. The definition of interpretation is provided in next paragraph and is the main definition to precisely define what is a satisfiable formula, and the set SAT of satisfiable formula, under a fixed propositional set of symbols Φ.

The constructive *i*nterpretation of IK is provided by a *s*tructure $\mathcal{I}$ formed by a non-empty set $\Delta^{\mathcal{I}}$ of worlds, or states, a (epistemic[3]) partial-order $\preceq^{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$, i.e., a reflexive, transitive and antisymmetric relation; a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and mapping $cdot^{\mathcal{I}}$ taking each atomic concept $p \in \Phi$ to a set $p^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ which is closed under $\preceq^{\mathcal{I}}$, i.e., $x \in p^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in p^{\mathcal{I}}$. In the literature on modal logic, what we have just defined is also called a (Kripke) model. We chose the term interpretation and this denotative definition by stylistic reasons. We also sometimes use structure as a synonym of interpretation, this use is emphatic when refering to the subjacent mathematical structure.

The *i*nterpretation $\mathcal{I}$ is extended from atomic concepts to arbitrary concepts via:

$$
\begin{aligned}
\top^{\mathcal{I}} &=_{df} \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} &=_{df} \emptyset \\
(\neg\varphi)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.x \preceq y \Rightarrow y \notin \varphi^{\mathcal{I}}\} \\
(\varphi_1 \wedge \varphi_2)^{\mathcal{I}} &=_{df} \varphi_1^{\mathcal{I}} \cap \varphi_2^{\mathcal{I}} \\
(\varphi_1 \vee \varphi_2)^{\mathcal{I}} &=_{df} \varphi_1^{\mathcal{I}} \cup \varphi_2^{\mathcal{I}} \\
(\varphi_1 \rightarrow \varphi_2)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.(x \preceq y \text{ and } y \in \varphi_1^{\mathcal{I}}) \Rightarrow y \in \varphi_2^{\mathcal{I}}\} \\
(\Diamond\varphi)^{\mathcal{I}} &=_{df} \{x \mid \exists y \in \Delta^{\mathcal{I}}.(x,y) \in R^{\mathcal{I}} \text{ and } y \in \varphi^{\mathcal{I}}\} \\
(\Box\varphi)^{\mathcal{I}} &=_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.x \preceq y \Rightarrow \forall z \in \Delta^{\mathcal{I}}.(y,z) \in R^{\mathcal{I}} \Rightarrow z \in \varphi^{\mathcal{I}}\}
\end{aligned}
$$

According to the semantics of IK, the structures $\mathcal{I}$ are models for IK whenever they satify two frame conditions:

F1 if $w \preceq w'$ and wRv then $\exists v'.w'Rv'$ and $v \preceq v'$

F2 if $v \preceq v'$ and wRv then $\exists w'.w'Rv'$ and $w \preceq w'$

The above conditions are diagrammatically expressed as:

$$
\begin{tikzcd}
w' \arrow[r, dotted, "R"] & v' \\
w \arrow[u, "\preceq"] \arrow[r, "R"'] \arrow[ur, phantom, "(F1)"] & v \arrow[u, dotted, "\preceq"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
w' \arrow[r, dotted, "R"] & v' \\
w \arrow[u, dotted, "\preceq"] \arrow[r, "R"'] \arrow[ur, phantom, "(F2)"] & v \arrow[u, "\preceq"']
\end{tikzcd}
$$

IK is simpler than [19] proposal of a constructive modal logic, since IK satisfies (like classical K) $\Diamond\bot = \bot$ and $\Diamond(\varphi_1 \vee \varphi_2) = \Diamond\varphi_1 \vee \Diamond\varphi_2$.

We call the reader's attention to note that, given an interpretation $\mathcal{I}$, $x \in \varphi^{\mathcal{I}}$ is equivalent to say that φ holds in the model $\mathcal{I}$ at the state x, or in usual notation $\mathcal{I}, x \models \varphi$. Using the above defined notion of *i*nterpretation we are in conditions to define satisfiable formulas in IK.

Definition 2 *A formula α is satisfiable in IK, iff, there is an interpretation $\mathcal{I} = \langle \Delta, \preceq, R, \cdot^{\mathcal{I}} \rangle$, such that, for each $w \in \Delta$, $\mathcal{I}, w \models \alpha$ (i.e. $w \in \alpha^{\mathcal{I}}$).*

Definition 3 *A set Γ of formulas is satisfiable, iff, there is an interpretation $\mathcal{I} = \langle \Delta, \preceq, R, \cdot^{\mathcal{I}} \rangle$, such that, for each $w \in \Delta$, for each $\gamma \in \Gamma$,$\mathcal{I}, w \models \gamma$.*

[3]Regarding the relationship between the worlds in a intuitionistic possible world model, the relation between worlds can be taken as the set of propositions that an hypothetical agent knows about the world

IK defines the usual (local) notion of logical consequence, that is complete and sound regarding the system in figure 1 [25].

Definition 4 *Let Γ be a set of IK formulas and α an IK formula. We say that α is an IK logical consequence of Γ, iff, for every interpretation $\mathcal{I} = \langle \Delta, \preceq, R, \cdot^{\mathcal{I}} \rangle$, $\forall w \in \Delta(\mathcal{I}, w \models \Gamma \Rightarrow \mathcal{I}, w \models \alpha)$.*

3 The Intuitionistic Hybrid Modal Logic IHML

A good way to improve the expressive power of a modal logic is to consider hybrid extensions of it. The fundamental resource that allows a logic to be called "hybrid" is a set of *nominals*. Nominals are a new kind of atomic symbol and they behave similarly to proposition symbols. The key difference between a nominal and a proposition symbol is related to their valuation in a model. While the set $\mathbf{V}(p)$ for a proposition symbol p can be any element of 2^V, the set $\mathbf{V}(i)$ for a nominal i has to be a singleton set. This way, each nominal is true at exactly one state (world) of the model, and thus, can be used to refer to this unique state. This is why these logics are called "hybrid": they are still modal logics, but they have the capacity to refer to specific states of the model, like in first-order logic. For a general introduction to hybrid logics, [2] and [5] can be consulted.

Definition 5 *Let us consider a hybrid language consisting of a set Φ of countably many proposition symbols (the elements of Φ are denoted by $p, q, \ldots$), a set Ω of countably many nominals (the elements of Ω are denoted by $i, j, \ldots$), such that $\Phi \cap \Omega = \emptyset$, the intuitionistic connectives $\neg, \vee, \rightarrow$ and $\wedge$, modal operators $\Diamond$ and $\Box$, and the operators $@_i$, for each nominal i (called* satisfaction operators*). The formulas are defined as follows:*

$$\varphi ::= p \mid i \mid \bot \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \rightarrow \varphi_2 \mid \Box \mid \Diamond\varphi \mid @_i\varphi$$

We freely use the standard abbreviation $\varphi_1 \leftrightarrow \varphi_2$ for $(\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)$

The definition of an interpretation for this language is the same as definition for ordinary (intuitionistic) modal logic, but the definition has to take care of nominals interpretation. Before introducing the precise definition, we have to comment that there seems not to be a definitive answer to the question: "What is the right Intuicionistic Hybrid Modal Logic" ? $IHML$ is built on top of IK and is a conservative extension. If we add the excluded middle to $IHML$ we obtain HML, the Hybrid Modal Logic (Classical).

Note that according to the definition of formulas $IHML$, the formula $@_j i$ is well-formed. What does it mean ? At world (state) j, the proposition i holds. Now, i is a proposition that holds only at the world j. In other words, $@_j i$ says that i and j are the "same" world. In this way, there must be a way to compare worlds, regarding some equivalence notion. At the same time, due to its intuitionistic feature, there is need to consider a local notion for this equivalence relation, since each nominal determinies a possible set of alternatives worlds. Thus, the semantics of $IHML$ has to include a component to denote this set of possible worlds, related to a given world. This semantics was initially proposed by Ewald [10] for intuitionistic tense-logic and adopted definitively for $IHML$ by Braüner and de Paiva in [6]. The following definition comes from [6].

Definition 6 *Let Φ be a set of propositions an $IHML$-model for Φ is $\mathcal{M} = \langle W, \preceq, \{\Delta_w\}_{w \in W}, \{\sim_w\}_{w \in W}, \{R_w\}_{w \in W}, \{\mathcal{V}_w\}_{w \in W} \rangle$, where :*

1. *W is the (non-empty) set of worlds, partially ordered by $\preceq$;*
2. *for each w, Δ_w is a non-empty set such that if $w \preceq v$ then $\Delta_w \subseteq \Delta_v$;*
3. *for each w, $\sim_w$ is an equivalence relation on Δ_w, such that, if $w \preceq v$ then $\sim_w \subseteq \sim_v$.*
4. *for each w, $R_w \subseteq \Delta_w \times \Delta_w$, such that if $w \preceq v$ then $R_w \subseteq R_v$;*
5. *for each w, $\mathcal{V}_w$ is a function from Φ to 2^{D_w}, such that, if $w \preceq v$ then $\mathcal{V}_w(p) \subseteq \mathcal{V}_v(p)$.*

Moreover, for each w and i, j, i', j', if $i \sim_w i'$, $j \sim_w j'$ and $iR_w j$ then $i'Rj'$. If $i \sim i'$ and $i \in \mathcal{V}_w(p)$ then $i' \in \mathcal{V}_w(p)$, for all $p \in \Phi$. This later condition together with 3, above, ensures that equivalent worlds must satisfy the same properties, The former condition together with 4 ensures that $\mathcal{M}$ is an IK model, taking into account only the modal fragment.

In order to interpret the Hybrid language into $\mathcal{M}$, we need to assign a unique world for each nominal n_i of the language. In order to simplify the reading of the following definition, we will considers that each nominal n_i is assigned to a unique world i, and all nominals in the language are distinct. Given a model $\mathcal{M}$, as above, the relation $\mathcal{M}, w, i \models \alpha$, where $w \in W$, $i \in \Delta_w$ and α is a formula on the propositional language Φ.

$\mathcal{M}, w, i \models p$ iff $i \in \mathcal{V}_w(p)$
$\mathcal{M}, w, i \models n_j$ iff $i \sim_w j$
$\mathcal{M}, w, i \models \alpha_1 \wedge \alpha_2$ iff $\mathcal{M}, w, i \models \alpha_1$ and $\mathcal{M}, w, i \models \alpha_2$
$\mathcal{M}, w, i \models \alpha_1 \vee \alpha_2$ iff $\mathcal{M}, w, i \models \alpha_1$ or $\mathcal{M}, w, i \models \alpha_2$
$\mathcal{M}, w, i \models \alpha_1 \rightarrow \alpha_2$ iff for all v, $w \preceq v$, if $\mathcal{M}, v, i \models \alpha_1$ then $\mathcal{M}, v, i \models \alpha_2$
$\mathcal{M}, w, i \models @_{n_j}\alpha$ iff $\mathcal{M}, w, j \models \alpha$
$\mathcal{M}, w, i \not\models \bot$
$\mathcal{M}, w, i \models \Diamond\alpha$ iff there is $k \in \Delta_w$, iR_wk and $\mathcal{M}, w, k \models \alpha$
$\mathcal{M}, w, i \models \Box\alpha$ iff for all v, $w \preceq v$, for all $k \in \Delta_v$, if iR_vk then $\mathcal{M}, v, k \models \alpha$

We say that $\mathcal{M}, w \models \alpha$ whenever $\mathcal{M}, w, i \models \alpha$ for every $i \in \Delta_w$. Analogously, $\mathcal{M} \models \alpha$ whenever $\mathcal{M}, w \models \alpha$, for every $w \in W$, under the supposition that the nominals n_i are interpreted uniquely in each Δ_w. Finally, a formula α is $IHML$-valid whenever $\mathcal{M} \models \alpha$, for every model $\mathcal{M}$. In [6] it is provided a sound and complete proof systems for IHML, regarded this semantics. It is known how one can translate modal formulas in first-order logic formulas preserving validity by means of the use of two-place relational symbols for R and one monadic symbol $A(x)$ for each propositional letter A in the modal language. In this way, the modal formula $\Box(A \wedge B)$ is translated in $\forall x(R(a,x) \rightarrow (A(x) \wedge B(x)))$, $\Diamond(B \rightarrow C)$ is translated to $\exists x(R(a,x) \wedge (B(x) \rightarrow C(x)))$, the nominal n_i is translated in the formula $a = i$ and the formula $@_{n_i}A$ is translated in $A(i)$, for example. In general, a formula α is translated into a formula $\alpha^\star$, such that, given a model $\mathcal{M}$ and a world $a \in W_{\mathcal{M}}$ it is the case that $\mathcal{M}, w \Vdash \alpha$ iff $\mathcal{M} \Vdash \alpha^\star(a)$ and a is interpreted as w. In this way, α is an $IHML$ tautology iff $\alpha^\star$ is a valid first-order intuitionistic formula (see [6] for a detailed discussion on this translation).

The translation, original from [6], maps $IHML$ formulas in a quite well-known fragment of first-order language, namely, the Guarded first-order logic with equality. The satisfability problem for this fragment of (classical) first-order logic is 2EXPTIME-complete. However, if we allow only two variables in the guards[4] the correponding SAT problem is only EXPTIME. These results can be found in [14]. As the translation to Intuitionistic first-order logic has to take into account the order relation regarded to the intuitionistic interpretation of the logical implication and the universal quantification, we have to consider a fragment of first-order (classical) logic that is able to express transitivity and reflexivity. It is shown in in [26, 17] that GF+TG, namely the guarded first-order logic that allows transitive relation only as guards and any relation, including the equality, elsewhere, is 2EXPTIME-hard. Thus, $IHML$ is 2EXPTIME-hard.

4 The Hybrid Logic IHK

As already shown, Hybrid logics add to usual modal logics a new kind of propositional symbols, the *nominals*, and also the so-called *satisfaction operators*. Because of the proximity of its corresponding description logic, namely $iALC$, we use here other notation for nominals, instead of @. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. If x is a nominal and X is an arbitrary formula, then a new formula $x:X$ called a satisfaction statement can be formed. The satisfaction statement $x:X$ expresses that the formula X is true at one particular world, namely the world denoted by x. In hindsight one can see that IHK shares with hybrid formalisms the idea of making the possible-world semantics part of the deductive system. While IK makes the relationship between worlds (e.g., xRy) part of the deductive system, IHK goes one step further and sees the worlds themselves x, y as part of the deductive system, (as they are now nominals) and the satisfaction relation itself as part of the deductive system, as it is now a syntactic operator, with modality-like properties. In contrast with the above mentioned approaches, ours assign a truth values to some formulas, also called assertions, they are not concepts as in [4], for example. Below we define the syntax of general assertions (A) and nominal assertions (N) for ABOX reasoning in IK.

[4] In the formulas $\forall x(R(a,x) \rightarrow (A(x) \wedge B(x)))$ and $\exists x(R(a,x) \wedge (B(x) \rightarrow C(x)))$, $R(a,x)$ is the guard.

Formulas (F) also includes subsumption of concepts interpreted as propositional statements.

$$N ::= x\colon C \mid x\colon N \qquad\qquad A ::= N \mid xRy \qquad\qquad F ::= A \mid C \sqsubseteq C$$

where x and y are nominals, R is a role symbol and C is a concept. In particular, this allows $x\colon (y\colon C)$, which is a perfectly valid nominal assertion.

Definition 7 (outer nominal) *In a nominal assertion $x\colon \gamma$, x is said to be the outer nominal of this assertion. That is, in an assertion of the form $x\colon (y\colon \gamma)$, x is the outer nominal.*

$\mathcal{I}, w \models C$ means $w \in C^{\mathcal{I}}$, that is, entity w satisfies concept C in the interpretation $\mathcal{I}$[5]. $\mathcal{I}$ is a model of C, written $\mathcal{I} \models C$ iff $\forall w \in \mathcal{I}.\mathcal{I}, w \models C$. $\models C$ denotes that $\forall \mathcal{I}.\mathcal{I} \models C$. All previous notions are extended to sets Φ of concepts in the usual way. $\mathcal{I}, w \models x\colon C$ holds, if and only if, $\forall z_x \succeq^{\mathcal{I}} x \,.\, \mathcal{I}, z_x \models C$. In a similar fashion, $\mathcal{I}, w \models xRy$ holds ,if and only if, $\forall z_x \succeq x.\forall z_y \succeq y.(x^{\mathcal{I}}_x, z^{\mathcal{I}}_y) \in R^{\mathcal{I}}$. That is, the evaluation of the hybrid formulas does not take into account only the world w, but it has to be monotonically preserved. It can be observed that for every w', if $x^{\mathcal{I}} \preceq w'$ and $\mathcal{I}, x' \models \alpha$, then $\mathcal{I}, w' \models \alpha$ holds.

Given a set Θ [6] of formulas and the set Γ of concepts, the following definition states when Θ, Γ entails δ.

Definition 8 *We write $\Theta, \Gamma \models \delta$ if it is the case that:*

$$\forall \mathcal{I}.((\forall x \in \Delta^{\mathcal{I}}.(\mathcal{I}, x \models \Theta)) \Rightarrow \forall (Nom(\Gamma, \delta)).\forall \vec{z} \succeq Nom(\Gamma, \delta).(\mathcal{I}, \vec{z} \models \Gamma \Rightarrow \mathcal{I}, \vec{z} \models \delta)$$

$\vec{z}$ is vector of variables $z_1, \dots, z_k$ and $Nom(\Gamma, \delta)$ is its vector of outer nominals occurrying in each nominal assertion of $\Gamma \cup \{\delta\}$. x is the only outer nominal of a nominal assertion $\{x\colon \gamma\}$, while a (pure) concept γ has no outer nominal.

$i\mathcal{ALC}$ arises from interpreting the usual possible worlds definitions in an intuitionistic meta-theory. As we already commented it is based on [6]. IHK is the hybrid logic associated to $i\mathcal{ALC}$İn the latter, concepts are described as:

$$C, D ::= A \mid \bot \mid \top \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid C \sqsubseteq D \mid \exists R.C \mid \forall R.C$$

In IHK concepts are taken as propositions and whenever the description logic semantics of a concept is a non-empty, its corresponding proposition holds in the related semantics. The reader can see the strong correspondence where C, D stands for concepts, A for an atomic concept, R for an atomic role.

$i\mathcal{ALC}$ syntax is more general than standard $\mathcal{ALC}$ since it includes subsumption $\sqsubseteq$ as a concept-forming operator. We have no use for nested subsumptions, but they do make the system easier to define, so we keep the general rules. Negation could be defined via subsumption, that is, $\neg C = C \sqsubseteq \bot$, but we find it convenient to keep it in the language. The constant $\top$ could also be omitted since it can be represented as $\neg\bot$. In IHK nested subsumptions, on the other hand, have the usual meaning assigned by the intuitionistic implication.

A constructive interpretation of $i\mathcal{ALC}$ is a structure $\mathcal{I}$ consisting of a non-empty set $\Delta^{\mathcal{I}}$ of entities in which each entity represents a partially defined individual; a refinement pre-ordering $\preceq^{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$, i.e., a reflexive and transitive relation; and an interpretation function $\cdot^{\mathcal{I}}$ mapping each role name R to a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and atomic concept A to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ which is closed under refinement, i.e., $x \in A^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in A^{\mathcal{I}}$. The interpretation $\mathcal{I}$ is lifted from atomic concepts to arbitrary concepts via:

$$\begin{array}{rl}
\top^{\mathcal{I}} & =_{df} \Delta^{\mathcal{I}} \\
\bot^{\mathcal{I}} & =_{df} \emptyset \\
(\neg C)^{\mathcal{I}} & =_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.x \preceq y \Rightarrow y \notin C^{\mathcal{I}}\} \\
(C \sqcap D)^{\mathcal{I}} & =_{df} C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} & =_{df} C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(C \sqsubseteq D)^{\mathcal{I}} & =_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.(x \preceq y \text{ and } y \in C^{\mathcal{I}}) \Rightarrow y \in D^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} & =_{df} \{x \mid \exists y \in \Delta^{\mathcal{I}}.(x, y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} & =_{df} \{x \mid \forall y \in \Delta^{\mathcal{I}}.x \preceq y \Rightarrow \forall z \in \Delta^{\mathcal{I}}.(y, z) \in R^{\mathcal{I}} \Rightarrow z \in C^{\mathcal{I}}\}
\end{array}$$

[5]In IHK, this w is a world and this satisfaction relation is possible world semantics

[6]Here we consider only acycled TBox with $\rightarrow$ and $\equiv$.

Interpretations $\mathcal{I}$ are models for $i\mathcal{ALC}$ if they satisfy two frame conditions 2 and 2 of section 2. Compared with the semantics of IK, the above semantics draws the conclusion that $\exists R.C$ could be read as $\Diamond C$ and $\forall R.C$ as $\Box C$. This in fact is the reason to consider $i\mathcal{ALC}$ as a multimodal version of IK without hybrid aspects. But we can say that TBOX reasoning is performed in multimodal IK.

Based on [23, 25, 13], the Hilbert calculus shown in Figure 1 implements TBox-reasoning. That is, it decides the semantical relationship $\Theta, \emptyset \models C$. Θ. This is shown in [9], as well as, a sequent calculus for ABOX reasoning.

In [3], a general approach to prove computational complexity of Hybrid Logics is presented. It is shown how to obtain, from a formula α, a 2-person game, designed to be polynomially implemented in an Alternating Turing Machine [21], such that, deciding existence of winning strategy for one of the players is equivalent to decide satisfiability (SAT) of α. This approach is used to show that SAT is in PSPACE, since any polynomial time implementation on an Alternating Turing Machine can be done in ordinary Turing Machine using polynomial space. Moreover, for Hybrid Modal Logic K it is possible to conclude PSPACE-completeness of SAT, since K is already PSPACE-complete.

Assertions like $a\colon C$, aRb and $a \preceq b$ are worth for ABox reasoning. In this complexity analysis of satisfiability in $i\mathcal{ALC}$ we consider this kind of assertions too. We prove that $i\mathcal{ALC}$, and hence IHK is PSPACE complete by adapting the game defined in [3] to our case. The game is a 2-person game of polynomial size on the size of the proposed formula and assertions (ABox). We consider the Hybrid assertions ($q\colon C$, aRb, $a \preceq b$). We admit that assertions like $a \preceq b$ might not be named as Hybrid, but they are formally treated as Hybrid in the approach. The lower bound is provided by the well-known theorem of Ladner [18] on PSPACE completeness of Intuitionistic Logic and the logics between K and S4.

Theorem 1 *$i\mathcal{ALC}$ is decidable regarding satisfiability. The complexity of satisfiability and derivability problems are PSPACE-complete.*

Proof 1 *The lower bound follows from the the fact that IPL is properly contained in $i\mathcal{ALC}$, and that IPL is PSPACE-complete. Consider a the (general) assertion $\Theta, \Gamma \sqsubseteq \gamma$, where Θ is the (sub)sequence of concept formulas and Γ is the (sub)sequence of assertion formulas, i.e, formulas either of the form qRp or $p : \alpha$. We have that $\Theta, \Gamma \Rightarrow \gamma$ is satisfiable, if and only if, $(\sqcap_{\theta\in\Theta}\theta) \sqsubseteq \gamma$ is satisfiable in a model of Γ. With the sake of a shorter presentation we consider only one role R. Let ξ be $(\sqcap_{\theta\in\Theta}\theta) \sqsubseteq \gamma$. If Δ is a set of formulas, a $\Delta^{\preceq}$ I-set is a maximal consistent set of subformulas from $\Delta \cup \{q \preceq p : NOMINALS(\Delta)\}$. The game is played as follows, by $\forall belard$ and $\exists loise$: on a list of $(\Gamma \cup \{\xi\})^{\preceq}$ I-sets. $\exists loise$ starts by playing a list $\{H_0, \ldots, H_k\}$ of $\Gamma \cup \{\xi\}$ I-sets, and two relations $\mathcal{R}$ and $\preccurlyeq$ on them. $\preccurlyeq$ is a pre-order relation on the I-sets.*

$\exists loise$ loses if one of the following conditions does not hold: [7]

CF1 *If $H_i \preccurlyeq H_m$ and $H_i\mathcal{R}H_j$ then there exists H_l, such that $H_m\mathcal{R}H_l$ and $H_j \preccurlyeq H_l$.*

CF2 *If $H_j \preccurlyeq H_l$ and $H_i\mathcal{R}H_j$ then there is H_m, such that $H_i \preccurlyeq H_m$ and $H_m\mathcal{R}H_l$.*

$\preceq$C *Let $\Sigma = \{p \preceq q : p \preceq q \in \Gamma\}$. Each H_i contains all the assertions representing the transitive-reflexive closure of Σ, under $\preceq$.*

NWI *H_0 contains $\Gamma \cup \{\xi\}$ and every other H_i contain at least one nominal occurring in $\Gamma \cup \{\xi\}$.*

NWII *No nominal occurs in more than one H_j, $j = 0, k$.*

NA *For every H_i and every $q : \alpha$ occurring in Γ, $q : \alpha \in H_i$, iff for some j $q \in H_j$ and $\alpha \in H_j$.*

DC *For all $\exists R.\alpha$ that is a subformula occurring in $\Gamma \cup \{\xi\}$, if H_iRH_j and $\exists R.\alpha \notin H_i$, then $\alpha \notin H_j$.*

ICI *For all $\neg\alpha$ that is a subformula occurring in $\Gamma \cup \{\xi\}$, if $H_i \preceq H_j$ and $\neg\alpha \notin H_i$, then $\alpha \in H_j$.*

ICII *For all $\alpha_1 \sqsubseteq \alpha_2$ that is a subformula occurring in $\Gamma \cup \{\xi\}$, if $H_i \preceq H_j$ and $\alpha_1 \sqsubseteq \alpha_2 \notin H_i$, then $\alpha_1 \in H_j$ and $\alpha_2 \notin H_j$*

[7] The labels of the items remind their logical roles in $i\mathcal{ALC}$ semantics: **N**aming**W**orlds I and II, **N**ominal**A**ssign, **D**iamond**C**ondition, **I**ntuitionistic**C**ondition I to III, **A**belard**D**iamond**C**ondition, **A**belard**I**ntuitionistic**C**ondition I and II, **I**nt**M**odelI and II, and $\preceq$**C**assertions

ICIII *For all* $q \preceq p$*, with* $q, p \in NOMINALS(\Gamma \cup \{\xi\})$*, if* $H_i \preccurlyeq H_j$*,* $q \in H_i$ *and* $p \in H_j$*, then* $q \preceq p \in H_n$*, for* $n = 0, \ldots, k$*.*

$\forall belard$ *continue by choosing an* H_i *and a formula* $\exists R.\alpha \in H_i$*, or* $\neg\alpha \in H_i$*, or* $\alpha_1 \sqsubseteq \alpha_2 \in H_i$*.* $\exists loise$ *must respond with an I-set* Y*, such that:*

ADC *If the chosen formula is* $\exists R\alpha$*, then* $\alpha \in Y$ *and for each subformula* $\exists R.\beta$ *from* $\Gamma \cup \{\xi\}$*, if* $\exists R.\beta \notin H_i$*, then* $\beta \notin Y$*.*

AICI *If the chosen formula is* $\neg\alpha$*, then* $\alpha \notin Y$ *and for each subformula* $\neg\beta$ *from* $\Gamma \cup \{\xi\}$*, if* $\neg\beta \notin H_i$*, then* $\beta \in Y$*. For each subformula* $\beta_1 \sqsubseteq \beta_2$ *from* $\Gamma \cup \{\xi\}$*, if* $\beta_1 \sqsubseteq \beta_2 \notin H_i$*, then* $\beta_1 \in Y$ *and* $\beta_2 \notin Y$*.*

AICII *If the chosen formula is* $\alpha_1 \sqsubseteq \alpha_2$*, then either* $\alpha_1 \in Y$ *and* $\alpha_2 \in Y$*, or* $\alpha_1 \notin Y$*. For each subformula* $\beta_1 \sqsubseteq \beta_2$ *from* $\Gamma \cup \{\xi\}$*, if* $\beta_1 \sqsubseteq \beta_2 \notin H_i$*, then* $\beta_1 \in Y$ *and* $\beta_2 \notin Y$*. For each subformula* $\neg\beta$ *from* $\Gamma \cup \{\xi\}$*, if* $\neg\beta \notin H_i$*, then* $\beta \in Y$*.*

IMI *In any case, for all* $q : \beta$ *that is a subformula of* $\Gamma \cup \{\xi\}$*,* $q : \beta \in Y$*, iff* $\{q, \beta\}$ *is contained in* H_j*, for some* $j = 0, k$*.*

IMII *If* $q \in Y$*, for some nominal* q*, then* $Y = H_j$ *for some* $j = 0, k$*. In this case* $\exists loise$ *wins the game.*

INeg *If* Y *is equal to some Hintikka I-set already generated by* $\exists loise$ *in a previous step of the game, then the game stops and she wins the game.*

The game stops and $\forall belard$ *wins, if* $\exists loise$ *cannot find an* Y *as above. If she can find such* Y*, it is added to the list of I-sets and the* $\preccurlyeq$*-relation is updated to* $\preccurlyeq \cup \{(H_i, Y)\}$ *and the match continues by* $\forall belard$ *choosing another formula from the recently updated list of Hintikka I-sets, considering the (possibly) updated* $\preccurlyeq$*-relation, leaving to her the task of finding another* Y*, and so on.*

At round m*,* $\forall belard$ *can only choose either a formula of modal depth less than or equal to the modal depth of* $\Gamma \cup \{\xi\}$ *minus* m*, or a formula with number of* $\neg$ *occurrences less than or equal to the* $\neg$ *occurrences of* $\Gamma \cup \{\xi\}$ *minus* m*. Finally,* $\exists loise$ *wins if she survive all attacks of* $\forall belard$*. Since each attack is performed on a formula of less or equal complexity than the last one, the maximum length of a match is bounded by the number of sub-formulas occurring in* $\Gamma \cup \{\xi\}$ *plus the number of nominals occurring in the original query, this is a polynomial bound on the length of the match, and hence the game. Using Lemma 1 we have that satisfiability of the sequent is equivalent to existence of a winning strategy for* $\exists loise$*. As existence of winning strategies is a PTIME decision problem in Alternating Turing Machines, we conclude that* $i\mathcal{ALC}$ *satisfiability is PSPACE-complete.*

Lemma 1 $\exists loise$ *has a winning strategy, if and only if* $\Gamma, \Theta \sqsubseteq \gamma$ *is satisfiable.*

Proof 2 *If* $\Gamma, \Theta \sqsubseteq \gamma$ *is satisfiable, then* $\Gamma \cup \{\xi\}$ *also is and the existence of a model that satisfies it allows the definition of an initial list of I-sets to* $\exists loise$ *play her winning strategy.* $\exists loise$ *has only to provide the I-sets associated to each world in this model of* $\Gamma \cup \{\xi\}$*. For the other direction, let us suppose that* $\exists loise$ *has a winning strategy. This winning strategy will provide us with model for* $\Gamma \cup \{\xi\}$*. Since* $\exists loise$ *has a winning strategy, she has answered to each possible move of* $\forall belard$*, she also has a winning starting list of I-sets. Thus,* $\exists loise$ *can produce a Hintikka I-set for each attack of her opponent. Let* M *be this collection of all Hintikka I-sets possible to be generated by the winning strategy of* $\exists loise$*. The model obtained is* $\langle M, R, \preccurlyeq, V\rangle$ *such that: (1) given I-sets* $M_i, M_j \in M$*,* $M_i R M_j$*, if and only if for every subformula* $\exists R.\beta$*, if* $\exists R.\beta \notin M_i$*, then* $\beta \notin M_j$*; (2)* $\preceq$ *is a relation on* M *obtained by* $\exists loise$ *using her winning strategy; (3)* $V(A) = \{M_i : A \in M_i\}$*; (4)* $q : \alpha$ *holds in* M_i*, iff,* $\{q, \alpha\} \subseteq M_i$*; (5)* $q \preceq p$ *holds in* M_i*, iff,* $H_j \preccurlyeq M_i$ *and* $q \preceq p \in H_j$*, for some* H_j *belonging to the initial I-sets provided by* $\exists loise$*. From Lemma 2 we can see that if* $\exists loise$ *has a winning strategy, then* $\Gamma \cup \{\xi\}$ *has a model. This finishes the remaining direction of this proof.*

Lemma 2 *For every subformula of* $\Gamma \cup \{\xi\}$*,* $\langle M, R, \preccurlyeq, V\rangle \models_{M_i} \alpha$*, if and only if,* $\alpha \in M_i$*.*

Proof 3 *This is proved by induction on the number of symbols in* α*.*

The following facts are used in the proof of Lemma 2.

Fact 1 *If* $M_i \preccurlyeq M_j$*, then for every subformula* $\alpha_1 \sqsubseteq \alpha_2$ *of* $\Gamma \cup \{\xi\}$*, if* $\alpha_1 \sqsubseteq \alpha_2 \notin M_i$*, then* $\alpha_1 \in M_j$ *and* $\alpha_2 \notin M_j$*.*

Fact 2 *If* $M_i \preccurlyeq M_j$*, then for every subformula* $\neg\alpha$ *of* $\Gamma \cup \{\xi\}$*, if* $\neg\alpha \notin M_i$*, then* $\alpha \in M_j$*.*

Fact 3 *If* $\alpha_1 \sqsubseteq \alpha_2 \in M_i$*, then for each* $Y \in M$*, such that,* $M_i \preccurlyeq Y$*, either* $\alpha_1 \in Y$ *and* $\alpha_2 \in Y$*, or* $\alpha_1 \notin Y$*.*

5 Conclusion

The main difference between $IHML$ and IHK ($i\mathcal{ALC}$) relies in the fact that the latter has only one fixed set of worlds that are the denotation of the nominals, while the former has one set of individuals for each world, and, these individuals are the denotation for the nominals. $i\mathcal{ALC}$ was designed with the special purpose of representing legal knowledge. The amount of individuals present in $IHML$ semantics was not useful for representing legal knowledge, according the jurisprudence principles discussed in [15], for example. From the fact that IHK is PSPACE-hard, we have as a corollary that IK is PSPACE-hard, and hence, complete, as well as IHK. Since $IHML$ is 2EXPTIME-hard, it is quite interesting to investigate what is the reason for this distance. We know, from the computational complexity literature, that the equality has a such strange consequence when included in a logical language. Sometimes it does not have any effect in the complexity and sometimes it turns the logic from decidable to undecidable. This is subject of further research,

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